**MATRIC NO:19/SCI01/090**

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1)

**Definition**

A *linear transformation* is a transformation T:Rn→Rm satisfying

T(u+v)=T(u)+T(v)T(cu)=cT(u)

for all vectors u,v in Rn and all scalars c.

**Facts about linear transformations**

*Let T:Rn→Rm be a linear transformation. Then:*

*T(0)=0.*

*For any vectors v1,v2,...,vk in Rn and scalars c1,c2,...,ck, we have*

*TAc1v1+c2v2+···+ckvkB=c1T(v1)+c2T(v2)+···+ckT(vk).*

**Examples**

**i)**

 Define T:R→R by T(x)=x+1. Is T a linear transformation?

Solution

We have T(0)=0+1=1. Since any linear transformation necessarily takes zero to zero by the above important note, we conclude that T is *not* linear (even though its graph is a line).

*Note:* in this case, it was not necessary to check explicitly that T does not satisfy both defining properties: since T(0)=0 is a consequence of these properties, at least one of them must not be satisfied. (In fact, this T satisfies neither.)

**ii)**

Define T:R2→R2 by T(x)=1.5x. Verify that T is linear.

Solution

We have to check the defining properties for *all* vectors u,v and *all* scalars c. In other words, we have to treat u,v, and c as *unknowns*. The only thing we are allowed to use is the definition of T.

T(u+v)=1.5(u+v)=1.5u+1.5v=T(u)+T(v)

Since T satisfies both defining properties, T is linear.

*Note:* we know from this example in Section 3.1 that T is a matrix transformation: in fact,

$T\left(x\right)=\left(\genfrac{}{}{0pt}{}{1.5 0}{0 1.5}\right)x$.

Since a matrix transformation is a linear transformation, this is another proof that T is linear.

**iii)**

Define T:R2→R3 by the formula

T$\left(\genfrac{}{}{0pt}{}{x}{y}\right)$=$\left(\genfrac{}{}{0pt}{}{3x-y}{\begin{array}{c}y\\x\end{array}}\right)$.

Verify that T is linear.

Solution

We have to check the defining properties for *all* vectors u,v and *all* scalars c. In other words, we have to treat u,v, and c as *unknowns*; the only thing we are allowed to use is the definition of T. Since T is defined in terms of the coordinates of u,v, we need to give those names as well; say u=$\left(\genfrac{}{}{0pt}{}{x1}{y2}\right)$ and v=$\left(\genfrac{}{}{0pt}{}{x2}{y2}\right)$. For the first property, we have

$$T\left(\left(\begin{array}{c}x1\\y1\end{array}\right)\right)+\left(\left(\genfrac{}{}{0pt}{}{x2}{y2}\right)\right)=T\left(\genfrac{}{}{0pt}{}{x1+x2}{y1+y2}\right)=\left(\genfrac{}{}{0pt}{}{\begin{array}{c}3\left(x1+x2\right)-\left(y1+y2\right)\\y1+y2\end{array}}{x1+x2}\right)$$

$$=\left(\genfrac{}{}{0pt}{}{\begin{array}{c}\left(3x1-y1\right)+\left(3x2-y2\right)\\y1+y2\end{array}}{x1+x2}\right)$$

$$=\left(\genfrac{}{}{0pt}{}{\begin{array}{c}3x1-y1\\y1\end{array}}{x1}\right)+\left(\genfrac{}{}{0pt}{}{\begin{array}{c}3x2-y2\\y2\end{array}}{x2}\right)=T\left(\genfrac{}{}{0pt}{}{x1}{y1}\right)+T\left(\genfrac{}{}{0pt}{}{x2}{y2}\right)$$

For the second property,

$$T\left(c\left(\begin{array}{c}x1\\y1\end{array}\right)\right)=T\left(\genfrac{}{}{0pt}{}{cx1}{cy1}\right)=\left(\genfrac{}{}{0pt}{}{3\left(cx1\right)-\left(cy1\right)}{\begin{array}{c}cy1\\cx1\end{array}}\right)$$

$$=\left(\genfrac{}{}{0pt}{}{\begin{array}{c}c\left(3x1-y1\right)\\cy1\end{array}}{cx1}\right)=c\left(\genfrac{}{}{0pt}{}{3x1-y1}{\begin{array}{c}y1\\x1\end{array}}\right)=cT\left(\genfrac{}{}{0pt}{}{x1}{y1}\right)$$

Since T satisfies the defining properties, T is a linear transformation.

*Note:* we will see in this example below that

$$T\left(\genfrac{}{}{0pt}{}{x}{y}\right)=\left(\genfrac{}{}{0pt}{}{3 -1}{\begin{array}{c}0 1\\1 0\end{array}}\right)\left(\genfrac{}{}{0pt}{}{x}{y}\right)$$

Hence T is in fact a matrix transformation.

**iv)**

 Verify that the following transformations from R2 to R2 are not linear:

T1$\left(\genfrac{}{}{0pt}{}{x}{y}\right)=\left(\genfrac{}{}{0pt}{}{|x|}{y}\right) T2\left(\genfrac{}{}{0pt}{}{x}{y}\right)=\left(\genfrac{}{}{0pt}{}{xy}{y}\right) T3\left(\genfrac{}{}{0pt}{}{x}{y}\right)=\left(\genfrac{}{}{0pt}{}{2x+1}{x-2y}\right)$

Solution

In order to verify that a transformation T is *not* linear, we have to show that T does not satisfy *at least one* of the two defining properties. For the first, the negation of the statement “T(u+v)=T(u)+T(v) for all vectors u,v” is “there exists at least one pair of vectors u,v such that T(u+v)A=T(u)+T(v).” In other words, it suffices to find *one example* of a pair of vectors u,v such that T(u+v)A=T(u)+T(v). Likewise, for the second, the negation of the statement “T(cu)=cT(u) for all vectors u and all scalars c” is “there exists some vector u and some scalar c such that T(cu)A=cT(u).” In other words, it suffices to find *one* vector u and *one* scalar c such that T(cu)A=cT(u).

For the first transformation, we note that

$$T1\left(-\left(\genfrac{}{}{0pt}{}{-1}{0}\right)\right)=T1\left(\genfrac{}{}{0pt}{}{-1}{0}\right)=\left(\genfrac{}{}{0pt}{}{|-1|}{0}\right)=\left(\genfrac{}{}{0pt}{}{1}{0}\right)$$

but that

$$-T1\left(\genfrac{}{}{0pt}{}{1}{0}\right)=-\left(\genfrac{}{}{0pt}{}{|1|}{0}\right)=-\left(\genfrac{}{}{0pt}{}{1}{0}\right)=\left(\genfrac{}{}{0pt}{}{-1}{0}\right)$$

Therefore, this transformation does not satisfy the second property.

For the second transformation, we note that

$$-T2\left(2\left(\genfrac{}{}{0pt}{}{1}{1}\right)\right)=T2\left(\genfrac{}{}{0pt}{}{2}{2}\right)=\left(\genfrac{}{}{0pt}{}{2.2}{2}\right)=\left(\genfrac{}{}{0pt}{}{4}{2}\right)$$

but that

$$2T2\left(\begin{array}{c}1\\1\end{array}\right)=2\left(\genfrac{}{}{0pt}{}{1.1}{1}\right)=2\left(\genfrac{}{}{0pt}{}{1}{1}\right)=\left(\genfrac{}{}{0pt}{}{2}{2}\right)$$

Therefore, this transformation does not satisfy the second property.

For the third transformation, we observe that

$$T3\left(\begin{array}{c}0\\0\end{array}\right)=\left(\genfrac{}{}{0pt}{}{2\left(0\right)+1}{0-2(0)}\right)=\left(\begin{array}{c}1\\0\end{array}\right)\ne \left(\genfrac{}{}{0pt}{}{0}{0}\right)$$

Since T3 does not take the zero vector to the zero vector, it cannot be linear.

**v)**

Define T:C3→C2T:C3→C2 by describing the output of the function for a generic input with the

 $T\left(\left[\genfrac{}{}{0pt}{}{\begin{array}{c}x1\\x2\end{array}}{x3}\right]\right)=\left[\begin{array}{c}2x1+x3\\-4x2\end{array}\right]$

and check the two defining properties.

$$T\left(x+y\right)=T\left(\left[\genfrac{}{}{0pt}{}{\begin{array}{c}x1\\x2\end{array}}{x3}\right]+\left[\genfrac{}{}{0pt}{}{\begin{array}{c}y1\\y2\end{array}}{y3}\right]\right)$$

$$T=\left(\left[\genfrac{}{}{0pt}{}{\begin{array}{c}x1+y1\\x2+y2\end{array}}{x3+y3}\right]\right)$$

$$\left[\begin{array}{c}2\left(x1+y1\right)+(x3+y3)\\-4(x2+y2)\end{array}\right]$$

$$\left[\begin{array}{c}\left(2x1+x3\right)+(2y1+y3)\\-4x2+\left(-4\right)y2\end{array}\right]$$

$$\left[\begin{array}{c}2x1+x3\\-4x2\end{array}\right]+\left[\begin{array}{c}2y1+y3\\-4y2\end{array}\right]$$

$$T\left(\left[\genfrac{}{}{0pt}{}{\begin{array}{c}x1\\x2\end{array}}{x3}\right]\right)+T\left(\left[\genfrac{}{}{0pt}{}{\begin{array}{c}y1\\y2\end{array}}{y3}\right]\right)$$

T(x)+T(y)

And

$$T\left(αx\right)=T\left(α\left[\genfrac{}{}{0pt}{}{\begin{array}{c}x1\\x2\end{array}}{x3}\right]\right)$$

$$=T\left(\left[\genfrac{}{}{0pt}{}{\begin{array}{c}x1\\x2\end{array}}{x3}\right]\right)$$

$$=\left[\begin{array}{c}2\left(αx1\right)+\left(αx3\right)\\-4\left(αx2\right)\end{array}\right]$$

$$=\left[\begin{array}{c}α\left(2x1+x3\right)\\α\left(-4x2\right)\end{array}\right]$$

$$=α\left[\begin{array}{c}2x1+x3\\-4x2\end{array}\right]$$

$$=αT\left(\left[\genfrac{}{}{0pt}{}{\begin{array}{c}x1\\x2\end{array}}{x3}\right]\right)$$

$$=αT(x)$$

T is a linear transformation.